

B.Sc. 2nd Semester
 STATISTICS (HONOURS)
 Paper: STA-HG-2016 (unit. 2)
Topic: Mathematical Expectation

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 21/05

Mathematical Expectation:

Definition: The mathematical expectation of a r.v. X , denoted by $E(X)$ is the sum of product of the values of the r.v. with the respective probabilities.

Thus, if X takes the values x_1, x_2, \dots, x_n with the respective probabilities p_1, p_2, \dots, p_n then

$$\begin{aligned} E(X) &= x_1 p_1 + x_2 p_2 + \dots + x_n p_n \\ &= \sum_{i=1}^n x_i p_i \quad \text{where } \sum_{i=1}^n p_i = 1 \end{aligned}$$

Again if the discrete r.v. X has pmf $f(x)$ then

$$E(X) = \sum_x x f(x) \quad \text{provided the series exists converges.}$$

For a continuous r.v. X following the pdf $f(x)$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx \quad \text{provided the integral converges absolutely}$$

$$\text{Here } \int_{-\infty}^{\infty} f(x) dx = 1.$$

Important remarks:

1. $E(X)$ exists if $E|X|$ exists.

2. The mathematical expectation of a r.v. X may not exist in some cases.

E.g. If X takes the values $0!, 1!, 2!, \dots$ with prob. ability law

$$P(X=x!) = \frac{e^{-1}}{x!}, \quad x=0, 1, 2, \dots$$

then

$$E(X) = \sum_{x=0}^{\infty} x! \cdot \frac{e^{-1}}{x!} = e^{-1} \sum_{x=0}^{\infty} 1 \quad \text{is infinite.}$$

Hence $E(X)$ does not exist in this case.

Mathematical Expectation of a function of a random variable:

If $g(x)$ is a function of a random variable X , then the mathematical expectation of $g(x)$ is defined as

$$E\{g(x)\} = \sum_x g(x) f(x) \quad \text{if } X \text{ is discrete with pmf } f(x)$$

$$= \int_{-\infty}^{\infty} g(x) f(x) dx \quad \text{if } X \text{ is continuous with pdf } f(x)$$

Particular cases:

1. If we put $g(x) = c$ then

$$E(c) = \sum_x c f(x) = c \sum_x f(x) = c \quad \because \sum_x f(x) = 1$$

Similarly for continuous r.v. X

$$E(c) = \int_{-\infty}^{\infty} c f(x) dx = c \int_{-\infty}^{\infty} f(x) dx = c \quad \because \int_{-\infty}^{\infty} f(x) dx = 1$$

Thus expectation of a constant is the constant ~~itself~~ itself.

2. If we take $g(x) = cx$

then $E(cx) = \sum_x cx f(x) = c \sum_x x f(x) = c E(x) \quad (\text{for discrete})$

or $E(cx) = \int_{-\infty}^{\infty} cx f(x) dx = c \int_{-\infty}^{\infty} x f(x) dx = c E(x) \quad (\text{for continuous})$

Hence

$$E(cx) = c E(x).$$

3. If $g(x) = x^r$, $r = 1, 2, 3, \dots$

then $E(x^r) = \int_{-\infty}^{\infty} x^r f(x) dx \quad (\text{where } X \text{ is assumed continuous})$

$$= \mu'_r \quad \mu'_r = \text{The } r\text{th moment about zero.}$$

$$r=1 \Rightarrow E(x) = \mu'_1 = \text{Mean.}$$

4. If we put $g(x) = \{x - E(x)\}^r = (x - \mu)^r$ where $\mu = E(x)$
then we have

$$E\{(x - \mu)^r\} = \int_{-\infty}^{\infty} (x - \mu)^r f(x) dx$$

$$= \mu_r \quad \text{where } \mu_r \text{ is the central moment.}$$

In particular if $r = 2$

$$E\{(x - \mu)^2\} = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

$$= \sigma^2, \text{ variance of } X.$$

Properties of Expectation:

(a) Additive theorem:

If X and Y are random variables then

$$E(X+Y) = E(X) + E(Y)$$

[Prove yourself separately for discrete and continuous X and Y]

In general if x_1, x_2, \dots, x_n are n random variables
then

$$E(x_1 + x_2 + \dots + x_n) = E(x_1) + E(x_2) + \dots + E(x_n)$$

$$\text{or } E\left(\sum_{i=1}^n x_i\right) = \sum_{i=1}^n E(x_i)$$

(b) Multiplicative theorem:

If X and Y are independent r.v. then

$$E(XY) = E(X)E(Y)$$

[Prove yourself separately for discrete and continuous r.v.]

In general for n independent r.v. x_1, x_2, \dots, x_n

$$E(x_1 x_2 \dots x_n) = E(x_1) \cdot E(x_2) \cdot \dots \cdot E(x_n)$$

$$\text{or } E\left(\prod_{i=1}^n x_i\right) = \prod_{i=1}^n E(x_i)$$

3. If X is a r.v. and a and b are constants then

$$E(ax + b) = aE(x) + b$$

Putting $a=1$ and $b=-\bar{x}$ we get

$$\begin{aligned} E(X - \bar{x}) &= E(X) - \bar{x} \\ &= \bar{x} - \bar{x} \\ &= 0 \end{aligned} \quad \therefore \bar{x} = E(X)$$

4. If x_1, x_2, \dots, x_n are r.v. and a_1, a_2, \dots, a_n are constants then

$$\begin{aligned} E(a_1x_1 + a_2x_2 + \dots + a_nx_n) &= a_1E(x_1) + a_2E(x_2) + \dots + a_nE(x_n) \\ \text{or } E\left(\sum_{i=1}^n a_i x_i\right) &= \sum_{i=1}^n a_i E(x_i) \end{aligned}$$

Definition: Variance: The variance of a random variable X denoted by $\text{Var}(X)$ or $V(X)$ is defined as

$$\begin{aligned} \text{Var}(X) &= E\{X - E(X)\}^2 \\ &= E(X^2) - \{E(X)\}^2 \quad (\text{By simplification}) \end{aligned}$$

provided $E(X)$ exists.

Defⁿ: Covariance: The covariance of two r.v. X and Y , denoted by $\text{Cov}(X, Y)$ is defined as.

$$\begin{aligned} \text{Cov}(X, Y) &= E[\{X - E(X)\}\{Y - E(Y)\}] \\ &= E(XY) - E(X)E(Y) \quad (\text{After simplification}) \end{aligned}$$

Thus

$\text{Var}(X) = E(X^2) - \{E(X)\}^2$
$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$

Some important results

1. If c is a constant then

$$\text{Var}(c) = 0$$

2. If x is r.v. and c is constant then

$$\text{Var}(cx) = c^2 \text{Var}(x)$$

3. If x and y are independent r.v. then

$$\begin{aligned} \text{Cov}(xy) &= E(xy) - E(x)E(y) \\ &= E(x)E(y) - E(x)E(y) \\ &= 0 \end{aligned}$$

4. If a and b are constants then

$$(i) \text{Var}(ax + b) = a^2 \text{Var}(x)$$

$$(ii) \text{Cov}(a, b) = 0$$

$$(iii) \text{Var}(ax \pm by) = a^2 \text{Var}(x) + b^2 \text{Var}(y) \pm 2ab \text{Cov}(x, y)$$

$$(iv) \text{Cov}(ax, by) = ab \text{Cov}(x, y)$$

(v) If x and y are independent then

$$\text{Var}(ax \pm by) = a^2 \text{Var}(x) + b^2 \text{Var}(y)$$

(vi) In general if x_1, x_2, \dots, x_n are n random variables

and a_1, a_2, \dots, a_n are constants then

$$\text{Var}\left(\sum_{i=1}^n a_i x_i\right) = \sum_{i=1}^n a_i^2 \text{Var}(x_i) + 2 \sum_{i=1}^n \sum_{j=1 \atop (i < j)}^n a_i a_j \text{Cov}(x_i, x_j)$$

$$= \sum_{i=1}^n a_i^2 \text{Var}(x_i) \quad \text{if } x_1, x_2, \dots, x_n \text{ are independent}$$